On uniformly bounded spherical functions in Hilbert space

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Abstract. Let G be a commutative group, written additively, with a neutral element 0, and let K be a finite group. Suppose that K acts on G via group automorphisms $G \ni a \mapsto ka \in G$, $k \in K$. Let $\mathfrak{H}$ be a complex Hilbert space and let $\mathscr{L}(\mathfrak{H})$ be the algebra of all bounded linear operators on $\mathfrak{H}$. A mapping $u\colon G \to \mathscr{L}(\mathfrak{H})$ is termed a K-spherical function if it satisfies (i) $|K|^{-1} \sum_{k \in K} u(a+kb) = u(a)u(b)$ for any $a, b \in G$, where $|K|$ denotes the cardinality of K, and (ii) $u(0) = \mathrm{id}_{\mathfrak{H}}$, where $\mathrm{id}_{\mathfrak{H}}$ designates the identity operator on $\mathfrak{H}$. The main result of the paper is that for each K-spherical function $u\colon G \to \mathscr{L}(\mathfrak{H})$ such that $\|u\|_\infty = \sup_{a \in G} \|u(a)\|_{\mathscr{L}(\mathfrak{H})} < \infty$, there is an invertible operator S in $\mathscr{L}(\mathfrak{H})$ with $\|S\| \, \|S^{-1}\| \leq |K| \, \|u\|_\infty^2$ such that the K-spherical function $\tilde{u}\colon G \to \mathscr{L}(\mathfrak{H})$ defined by $\tilde{u}(a) = Su(a)S^{-1}$, $a \in G$, satisfies $\tilde{u}(-a) = \tilde{u}(a)^*$ for each $a \in G$. It is shown that this last condition is equivalent to insisting that $\tilde{u}(a)$ be normal for each $a \in G$.

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1. Introduction

Let G be a commutative group, written additively, with a neutral element 0. Let K be a finite group, written multiplicatively, with a neutral element e. Suppose that G is a K-space under a map $K \times G \ni (k, a) \mapsto ka \in G$ satisfying the following conditions:

(i) $k(la) = (kl)a$ for all $k, l \in K$ and all $a \in G$;
(ii) $ea = a$ for all $a \in G$;
(iii) $k(a + b) = ka + kb$ for all $k \in K$ and all $a, b \in G$;

Condition (iii) automatically implies, as is easily seen, two more conditions:

(iv) $k0 = 0$ for all $k \in K$;

(v) $k(-a) = -ka$ for all $k \in K$ and all $a \in G$.

Thus each map $G \ni a \mapsto ka \in G$, $k \in K$, is a group automorphism, and all the conditions (i) to (v) can be rephrased as saying that K acts on G via group automorphisms of G.

Let $\mathfrak{H}$ be a complex Hilbert space. Let $\langle \cdot, \cdot \rangle$ be the inner product in $\mathfrak{H}$ and let $\| \cdot \|$ be the norm induced by this inner product. Let $\mathscr{L}(\mathfrak{H})$ be the algebra of all bounded linear operators on $\mathfrak{H}$ with the usual operator norm denoted $\| \cdot \|_{\mathscr{L}(\mathfrak{H})}$ or, briefly, $\| \cdot \|$. Designate by $\mathrm{id}_{\mathfrak{H}}$ the identity operator on $\mathfrak{H}$. Given a set A, denote by $|A|$ the cardinality of A.

In accordance with the terminology employed by Stetkær [20], a K-*spherical function* on G with values in $\mathscr{L}(\mathfrak{H})$ is a mapping $u \colon G \to \mathscr{L}(\mathfrak{H})$ such that

$$\frac{1}{|K|} \sum_{k \in K} u(a + kb) = u(a)u(b) \quad (a, b \in G),$$
$$u(0) = \mathrm{id}_{\mathfrak{H}} .$$

Basic examples of spherical functions include *group representations* and *cosine functions*. Spherical functions of both such types can be defined on an arbitrary commutative group. Denote by $\mathbb{Z}_k$ the cyclic group of order k. A representation π of G in $\mathscr{L}(\mathfrak{H})$ is a mapping $\pi \colon G \to \mathscr{L}(\mathfrak{H})$ satisfying

$$\pi(a + b) = \pi(a)\pi(b) \quad (a \in G),$$
$$\pi(0) = \mathrm{id}_{\mathfrak{H}} .$$

It is a $\mathbb{Z}_1$-spherical function when G is treated as a $\mathbb{Z}_1$-space under the trivial action $ea = a$, $a \in G$, where e denotes the sole, neutral element of $\mathbb{Z}_1$. A cosine function c on G is a mapping $c \colon G \to \mathscr{L}(\mathfrak{H})$ satisfying

$$c(a + b) + c(a - b) = 2c(a)c(b) \quad (a, b \in G),$$
$$c(0) = \mathrm{id}_{\mathfrak{H}} .$$

It is a $\mathbb{Z}_2$-spherical function when G is considered as a $\mathbb{Z}_2$-space under the action

$$ea = a, \quad \jmath a = -a \quad (a \in G).$$

Here $\jmath$ is the unique order-2 element of $\mathbb{Z}_2$ and e is the neutral element of $\mathbb{Z}_2$. More involved spherical functions can be viewed as "higher-order" analogues of group representations and cosine functions. These are defined on commutative groups that are K-spaces for more complex groups K. One example of a commutative group with a non-trivial K-space structure is the additive group of complex numbers $\mathbb{C}$. For each $k \in \mathbb{N}$, $\mathbb{C}$ can be turned into a $\mathbb{Z}_k$-space with the aid of the mapping

$$\mathbb{Z}_k \times \mathbb{C} \ni (k, z) \mapsto \varepsilon^k z \in \mathbb{C}.$$

Here ε is a primitive kth root of unity, and $\mathbb{Z}_k$ is identified with the additive group of classes of integers modulo k.

Let $l^\infty(G, \mathscr{L}(\mathfrak{H}))$ be the Banach space of all functions $f \colon G \to \mathscr{L}(\mathfrak{H})$ such that $\|f\|_\infty = \sup_{a \in G} \|f(a)\|_{\mathscr{L}(\mathfrak{H})} < \infty$. The members of $l^\infty(G, \mathscr{L}(\mathfrak{H}))$

will be termed *uniformly bounded* $\mathscr{L}(\mathfrak{H})$-valued functions on G. Given an operator T in $\mathscr{L}(\mathfrak{H})$, denote by T^* the adjoint of T. For an invertible operator S in $\mathscr{L}(\mathfrak{H})$, denote by $\kappa(S)$ the *condition number* of S defined as $\|S\|\,\|S^{-1}\|$.

The aim of this paper is to establish the following.

Theorem 1. *Let $u\colon G \to \mathscr{L}(\mathfrak{H})$ be a uniformly bounded K-spherical function. Then there is an invertible operator S in $\mathscr{L}(\mathfrak{H})$ with*

$$\kappa(S) \le |K|\,\|u\|_\infty^2 \tag{1.1}$$

such that the K-spherical function $\tilde{u}\colon G \to \mathscr{L}(\mathfrak{H})$ defined by $\tilde{u}(a) = Su(a)S^{-1}$, $a \in G$, satisfies $\tilde{u}(-a) = \tilde{u}(a)^*$ for each $a \in G$.

In the sequel, a mapping $u\colon G \to \mathscr{L}(\mathfrak{H})$ satisfying $u(-a) = u(a)^*$ for each $a \in G$ will be referred to as a $*$-*mapping*. An invertible operator S in $\mathscr{L}(\mathfrak{H})$ is usually called a *similarity*. Two operators T_1 and T_2 on $\mathfrak{H}$ are said to be *similar* if there is a similarity S such that $T_2 = ST_1S^{-1}$. Two functions $f\colon X \to \mathscr{L}(\mathfrak{H})$ and $g\colon X \to \mathscr{L}(\mathfrak{H})$, where X is a set, are called similar if there is a similarity S such that $g(x) = Sf(x)S^{-1}$ for each $x \in X$. With this terminology, Theorem 1 can be rephrased as saying that any uniformly bounded K-spherical function from G into $\mathscr{L}(\mathfrak{H})$, where $\mathfrak{H}$ is a Hilbert space, is similar to a $*$-K-spherical function, with the underlying similarity satisfying (1.1).

It is evident that the term $*$-group representation is synonymous with *unitary* group representation. Recall that a representation $\pi\colon G \to \mathscr{L}(\mathfrak{H})$ is unitary if $\pi(a)^{-1} = \pi(a)^*$ for every $a \in G$. Similarly, the term $*$-cosine function is equivalent to *hermitian* cosine function. This follows from the fact that if c is a cosine function, then $c(a) = c(-a)$ for each $a \in G$ (see Proposition 1 below). As one might expect, a mapping $f\colon X \to \mathscr{L}(\mathfrak{H})$ is called hermitian if $f(x) = f(x)^*$ for every $x \in X$. Now it emerges that, generally, a uniformly bounded spherical function is $*$-spherical if and only if it is *normal* (see Propositions 3 and 5). By definition, a mapping $f\colon X \to \mathscr{L}(\mathfrak{H})$ is normal if $f(x)$ is normal for each $x \in X$. Thus Theorem 1 can be equivalently restated as follows.

Theorem 2. *Let $u\colon G \to \mathscr{L}(\mathfrak{H})$ be a uniformly bounded K-spherical function. Then there is an invertible operator S in $\mathscr{L}(\mathfrak{H})$ with $\kappa(S) \le |K|\,\|u\|_\infty^2$ such that the K-spherical function $\tilde{u}\colon G \to \mathscr{L}(\mathfrak{H})$ defined by $\tilde{u}(a) = Su(a)S^{-1}$, $a \in G$, is normal.*

Interestingly, Theorem 1 can also be viewed as a result about renorming. This follows from the fact that any uniformly bounded $*$-K-spherical function $u\colon G \to \mathscr{L}(\mathfrak{H})$ is contractive with bound $\|u\|_\infty = 1$ (see Proposition 4 below). Theorem 1 ensures that any uniformly bounded K-spherical function from G into $\mathscr{L}(\mathfrak{H})$ is similar to a contractive one.

Theorem 1 has several predecessors in the literature and is representative of a class of results concerning *similarity problems* [14]. Recall that a group G is *unitarisable* if every uniformly bounded representation of G on a Hilbert space is similar to a unitary representation. Sz.-Nagy [22] proved that $\mathbb{Z}$ is unitarisable. Dixmier [7] and Day [6] extended this result to all amenable (discrete) groups. Specialised to group representations, Theorem 1 recovers Dixmier and Day's result in the case of commutative groups, every such group being amenable. The bound for the underlying similarity obtained in the classical proof of Dixmier and Day's result coincides with that given in (1.1) with $|K| = 1$. In the context of cosine functions, Theorem 1 ensures that any uniformly bounded cosine function on a commutative group G with values in $\mathscr{L}(\mathfrak{H})$, where $\mathfrak{H}$ is a Hilbert space, is similar to a hermitian cosine function. This result was first proved by Fattorini [8] in the case $G = \mathbb{R}$. Kurepa [12] extended it, effectively, to the case when $G^{(2)} = \{a \in G \mid a = 2b \text{ for } b \in G\}$ is a subgroup of G of finite index (this includes such groups like a direct sum of finitely many copies of the group of integers $\mathbb{Z}$, but excludes such groups like a direct sum or a direct product of countably many copies of $\mathbb{Z}$). Finally, the author [3] extended Kurepa's result to the case of an arbitrary commutative group G, obtaining, however, a bound for the condition number of the underlying similarity weaker than (1.1), namely $\kappa(S) \leq 2(1 + 2\|u\|_\infty)^3 \, \|u\|_\infty^5$. Considering more general spherical functions, Stetkær [21] proved Theorem 1 under the extra condition that, for each $k \in K \setminus \{e\}$, the homomorphism $G \ni a \mapsto a - ka \in G$ is surjective, obtaining again a bound for the condition number of the underlying similarity weaker than (1.1), namely $\kappa(S) \leq |K|^2 \, \|u\|_\infty^2$. Incidentally, for cosine functions, Stetkær's result is subsumed by that of Kurepa, as then the extra condition just mentioned reduces to the requirement that $G = G^{(2)}$. The essence of our contribution is that Theorem 1 holds true without any additional conditions on G and K, with a sharper bound on the condition number of the underlying similarity than the similar bounds obtained previously.

2. Basic facts

We begin by establishing a few results concerning spherical functions, of which some were already alluded to in the Introduction. From now on, G will always denote a commutative group and K will be a finite group. We shall assume that K acts on G via group automorphisms $G \ni a \mapsto ka \in G$, $k \in K$. The symbol $\mathfrak{H}$ will denote a complex Hilbert space.

Proposition 1. *Let $u\colon G \to \mathscr{L}(\mathfrak{H})$ be a K-spherical function. Then $u(ka) = u(a)$ for each $a \in G$ and each $k \in K$.*

Proof. For each $a \in G$, we have

$$u(a) = u(0)u(a) = \frac{1}{|K|} \sum_{l \in K} u(la).$$

Hence, for each $k \in K$,

$$u(ka) = \frac{1}{|K|} \sum_{l \in K} u(lka) = \frac{1}{|K|} \sum_{l \in K} u(la) = u(a).$$

□

Proposition 2. *Let $u\colon G \to \mathscr{L}(\mathfrak{H})$ be a K-spherical function. Then the family $\{u(a) \mid a \in G\}$ is commutative.*

Proof. In view of Proposition 1, for any $a, b \in G$,

$$u(a)u(b) = \frac{1}{|K|}\sum_{l\in K} u(a+lb) = \frac{1}{|K|^2}\sum_{k\in K}\left[\sum_{l\in K} u(k(a+lb))\right]$$
$$= \frac{1}{|K|^2}\sum_{k\in K}\left[\sum_{l\in K} u(ka+klb)\right] = \frac{1}{|K|^2}\sum_{k,l\in K} u(ka+lb).$$

The rightmost sum above remains intact if a and b are interchanged. Consequently, $u(a)$ and $u(b)$ commute. □

Proposition 3. *Every $*$-K-spherical function on G with values in $\mathscr{L}(\mathfrak{H})$ is normal.*

Proof. Let $u\colon G \to \mathscr{L}(\mathfrak{H})$ be a $*$-K-spherical function. By Proposition 2, for each $a \in G$, $u(a)$ and $u(-a)$ commute. But, by assumption, $u(-a) = u(a)^*$, so effectively $u(a)$ and $u(a)^*$ commute. □

Proposition 4. *Let $u\colon G \to \mathscr{L}(\mathfrak{H})$ be a uniformly bounded $*$-K-spherical function. Then $\|u\|_\infty = 1$.*

Proof. Since $\|u(0)\| = 1$, all we need to show is that $\|u\|_\infty$ does not exceed 1. Now, for any $a \in G$,

$$\|u(a)\|^2 = \|u(a)u(a)^*\| = \|u(a)u(-a)\|$$
$$= \left\|\frac{1}{|K|}\sum_{k\in K} u(a-ka)\right\| \le \frac{1}{|K|}\sum_{k\in K}\|u(a-ka)\| \le \|u\|_\infty.$$

Hence $\|u\|_\infty^2 \le \|u\|_\infty$ and further $\|u\|_\infty \le 1$, as desired. □

For a locally compact commutative group G, denote by $\hat{G}$ the dual group of G. For $\chi \in \hat{G}$ and $a \in G$, let (a, χ) denote the value of χ at a. For a complex unital commutative Banach algebra $\boldsymbol{A}$, let $\Delta(\boldsymbol{A})$ be the Gelfand space of $\boldsymbol{A}$, i.e., the set of all unital homomorphisms from $\boldsymbol{A}$ to $\mathbb{C}$.

We now pass to proving a converse to Proposition 3. The proof will employ the following characterisation of scalar bounded spherical functions established in [4] (see also [1, 19]):

Theorem 3. *Suppose that G is a locally compact commutative group. Then, for each $\chi \in \hat{G}$, the mapping $u\colon G \to \mathbb{C}$ given by*

$$u(a) = \frac{1}{|K|}\sum_{k\in K}(ka, \chi) \quad (a \in G) \tag{2.1}$$

is a bounded continuous K-spherical function. Conversely, every bounded continuous K-spherical function $u\colon G \to \mathbb{C}$ can be expressed as in (2.1) *for some $\chi \in \hat{G}$.*

Proposition 5. *Every uniformly bounded normal K-spherical function on G with values in $\mathscr{L}(\mathfrak{H})$ is $*$-K-spherical.*

Proof. Let $u\colon G \to \mathscr{L}(\mathfrak{H})$ be a bounded normal K-spherical function. Let $\boldsymbol{C}$ be the C^*-algebra subalgebra of $\mathscr{L}(\mathfrak{H})$ generated by $\{u(a) \mid a \in G\}$. In view of the Fuglede–Putnam–Rosenblum theorem [9,15,16] (see also [17, Theorem 12.16]), $\boldsymbol{C}$ is commutative. Since $\boldsymbol{C}$ is symmetric and semi-simple, it suffices to show that $\phi(u(-a)) = \overline{\phi(u(a))}$ for each $\phi \in \Delta(\boldsymbol{C})$ and each $a \in G$. Clearly, the mapping $a \mapsto \phi(u(a))$ is a bounded K-spherical function on G. Endow G with the discrete topology, making G into a locally compact group. In view of Theorem 3, there exists $\chi \in \hat{G}$ such that

$$\phi(u(a)) = \frac{1}{|K|} \sum_{k \in K} (ka, \chi)$$

for each $a \in G$. Now

$$\begin{aligned}\phi(u(-a)) &= \frac{1}{|K|} \sum_{k \in K} (k(-a), \chi) = \frac{1}{|K|} \sum_{k \in K} (-ka, \chi) \\ &= \frac{1}{|K|} \sum_{k \in K} \overline{(ka, \chi)} = \overline{\phi(u(a))}\end{aligned}$$

for each $a \in G$, as was to be shown. □

3. Miscellaneous results

This section provides preparatory material needed for the proof of the main result.

3.1. Invariant means

Let $l^\infty(G)$ be the Banach space of all functions $f\colon G \to \mathbb{C}$ such that $\|f\|_\infty = \sup_{a \in G} |f(a)| < \infty$. For $a \in G$, denote by T_a the operator of translation by a defined by $(T_a f)(b) = f(a+b)$, $b \in G$. Let m be an invariant (or Banach) mean on $l^\infty(G)$, that is, a bounded linear functional on $l^\infty(G)$ satisfying the following conditions:

(i) $\|m\| = 1 = m(1)$;
(ii) $m(T_a f) = m(f)$ for each $f \in l^\infty(G)$ and each $a \in G$.

Then, as a familiar argument (see [18, p. 109]) shows, (i) implies

(iii) $m(f) \geq 0$ for each $f \in l^\infty(G)$ such that $f \geq 0$.

The existence of m is ensured by a theorem of Day [5] (see also [11, §17.5] and [10, Theorem 1.2.1]). Corresponding to m, we define an operator-valued invariant mean M on $l^\infty(G, \mathscr{L}(\mathfrak{H}))$ as follows. If $f \in l^\infty(G, \mathscr{L}(\mathfrak{H}))$ and $x, y \in \mathfrak{H}$, then $a \mapsto \langle f(a)x, y\rangle$ is a function in $l^\infty(G)$ with norm at most $\|f\|_\infty \|\|x\|\|y\|$. Thus it makes sense to apply the invariant mean m to the function $a \mapsto \langle f(a)x, y\rangle$ to get $m_a(\langle f(a)x, y\rangle)$; here the subscript a indicates that the mean is taken with respect to the variable a. It is easily verified that

$$(x, y) \mapsto m_a(\langle f(a)x, y\rangle)$$

is a bounded sesquilinear form on $\mathfrak{H} \times \mathfrak{H}$ with bound $\|f\|_\infty$, so, by the Riesz representation theorem, there is an operator $M(f)$ in $\mathscr{L}(\mathfrak{H})$ with $\|M(f)\| \leq \|f\|_\infty$ such that

$$\langle M(f)x, y\rangle = m_a(\langle f(a)x, y\rangle)$$

for all $x, y \in \mathfrak{H}$. Furthermore, it is easily seen that

$$M\colon l^\infty(G, \mathscr{L}(\mathfrak{H})) \to \mathscr{L}(\mathfrak{H}), \quad f \mapsto M(f),$$

is a bounded linear operator satisfying

(i) $\|M\| = 1$;
(ii) for each $f \in l^\infty(G, \mathscr{L}(\mathfrak{H}))$, $M(f)$ is in the weak-operator closure of the convex hull of $\{f(a) \mid a \in G\}$;
(iii) $M(c_A) = A$ for each $A \in \mathscr{L}(\mathfrak{H})$, where c_A denotes the constant function taking value A;
(iv) $M(T_a f) = M(f)$ for each $f \in l^\infty(G, \mathscr{L}(\mathfrak{H}))$ and each $a \in G$.

Note that statement (iii) is an immediate consequence of statement (ii).

Let H be a commutative group and let n be invariant mean on $l^\infty(H)$. If $f \in l^\infty(G \times H)$, then $\sup_{b \in G} |m_a(f(a,b))| \leq \|f\|_\infty$ so $b \mapsto m_a(f(a,b))$ is a function in $l^\infty(H)$ and one can apply n to it to get $n_b(m_a(f(a,b)))$. Setting

$$(m \otimes n)(f) = n_b(m_a(f(a,b))) \quad (f \in l^\infty(G \times H))$$

defines a linear bounded functional $m \otimes n$ on $l^\infty(G \times H)$. This will be termed the *tensor product* of m and n. It is readily seen that $m \otimes n$ is an invariant mean on $l^\infty(G \times H)$. A related notion is that of a *tensor power* of an invariant mean. For each $n \in \mathbb{N}$, the nth tensor power of m is the invariant mean on $l^\infty(G^n)$ defined inductively by the rule

$$m^{\otimes 1} = m, \quad m^{\otimes n} = m \otimes m^{\otimes (n-1)}.$$

Clearly,

$$m^{\otimes n}(f) = m_{a_n}(m_{a_{n-1}}(\ldots(m_{a_1}(f(a_1, \ldots, a_n)))\ldots))$$

for each $f \in l^\infty(G^n)$. Sometimes we shall write $m^{\otimes n}_{(a_1,\ldots,a_n)}(f(a_1, \ldots, a_n))$ to denote $m^{\otimes n}(f)$.

Let M, N, and $M \otimes N$ be the invariant means on $l^\infty(G, \mathscr{L}(\mathfrak{H}))$, $l^\infty(H, \mathscr{L}(\mathfrak{H}))$, and $l^\infty(G \times H, \mathscr{L}(\mathfrak{H}))$ induced by m, n, and $m \otimes n$, respectively.

Lemma 1. *If $f \in l^\infty(G \times H, \mathscr{L}(\mathfrak{H}))$, then the function $b \mapsto M_a(f(a,b))$ is in $l^\infty(H, \mathscr{L}(\mathfrak{H}))$ and*

$$(M \otimes N)(f) = N_b(M_a(f(a,b))).$$

Proof. Let $f \in l^\infty(G \times H, \mathscr{L}(\mathfrak{H}))$. Clearly, $\sup_{b \in G} \|M_a(f(a,b))\| \leq \|f\|_\infty$ and, for any $x, y \in \mathfrak{H}$,

$$\begin{aligned}
\big\langle (M \otimes N)(f)x, y\big\rangle &= n_b(m_a(\langle f(a,b)x, y\rangle)) \\
&= n_b(\langle M_a(f(a,b))x, y\rangle) \\
&= \langle N_b(M_a(f(a,b)))x, y\rangle,
\end{aligned}$$

which establishes the assertion. $\square$

For $f\colon G \to \mathscr{L}(\mathfrak{H})$ and $g\colon H \to \mathscr{L}(\mathfrak{H})$, we define $f \odot g\colon G \times H \to \mathscr{L}(\mathfrak{H})$ and $f \odot_r g\colon G \times H \to \mathscr{L}(\mathfrak{H})$ by

$$(f \odot g)(a,b) = f(a)g(b),$$
$$(f \odot_r g)(a,b) = g(b)f(a)$$

for each $a \in G$ and each $b \in H$.

Lemma 2. *Let f be a function in $l^\infty(G, \mathscr{L}(\mathfrak{H}))$ and g be a function in $l^\infty(H, \mathscr{L}(\mathfrak{H}))$. Then*

(i) $M(f)N(g) = (M \otimes N)(f \odot g)$;
(ii) $M(f)N(g) = (N \otimes M)(g \odot_r f)$;
(iii) *if $f(a)$ and $g(b)$ commute for any $a \in G$ and any $b \in H$, then $M(f)$ commutes with $N(g)$.*

Proof. For any $a \in G$, any $b \in H$, and any $x, y \in \mathfrak{H}$,

$$\begin{aligned}
\langle M_a(f(a))N_b(g(b))x, y\rangle &= \langle N_b(g(b))x, M_a(f(a))^* y\rangle \\
&= n_b(\langle g(b)x, M_a(f(a))^* y\rangle) \\
&= n_b(\langle M_a(f(a))g(b)x, y\rangle) \\
&= n_b(m_a(\langle f(a)g(b)x, y\rangle)) \\
&= n_b(m_a(\langle (f \odot g)(a,b)x, y\rangle)) \\
&= \langle (M \otimes N)(f \odot g)x, y\rangle.
\end{aligned}$$

This proves (i).

Likewise, for any $a \in G$, any $b \in H$, and any $x, y \in \mathfrak{H}$,

$$\begin{aligned}
\langle M_a(f(a))N_b(g(b))x, y\rangle &= m_a(\langle f(a)N_b(g(b))x, y\rangle) \\
&= m_a(\langle N_b(g(b))x, f(a)^* y\rangle) \\
&= m_a(n_b(\langle g(b)x, f(a)^* y\rangle)) \\
&= m_a(n_b(\langle f(a)g(b)x, y\rangle)) \\
&= m_a(n_b(\langle (g \odot_r f)(b,a)x, y\rangle)) \\
&= \langle (N \otimes M)(g \odot_r f)x, y\rangle.
\end{aligned}$$

This gives (ii).

If $f(a)$ and $g(b)$ commute for any $a \in G$ and any $b \in H$, then $g \odot f = g \odot_r f$, and so, by parts (i) and (ii) already established,

$$\begin{aligned}
M(f)N(g) &= (N \otimes M)(g \odot_r f) \\
&= (N \otimes M)(g \odot f) \\
&= N(g)M(f).
\end{aligned}$$

This completes part (iii). □

3.2. Selector subsequences

Let $(k_i)_{i=1}^I$ be a finite sequence with values in K. Corresponding to this sequence, we define a subsequence $(k_{i_j})_{j=1}^J$ by imposing the following conditions:

(i) $\{k_1, \dots, k_I\} = \{k_{i_1}, \dots, k_{i_J}\}$,
(ii) if $k_{i_j} = k_k$, then $i_j \leq k$.

In other words, the subsequence in question has the same range as the initial sequence and attains each admissible value at a lowest possible index. Henceforth $(k_{i_j})_{j=1}^J$ will be referred to as the selector subsequence associated with $(k_i)_{i=1}^I$. The construction of $(k_{i_j})_{j=1}^J$ proceeds on a "first-come, first-taken" basis. We let $i_1 = 1$ and take for i_2 the smallest i such that $k_i \in \{k_1, \dots, k_I\} \setminus \{k_{i_1}\}$. If $k_{i_1}, \dots, k_{i_j}$ have been already picked, we take for i_{j+1} the smallest i such that $k_i \in \{k_1, \dots, k_I\} \setminus \{k_{i_1}, \dots, k_{i_j}\}$. Continuing in this way, we eventually exhaust all possible range values for $(k_i)_{i=1}^I$. The resulting subsequence clearly possesses the required properties. The rationale for introducing the concept of selector subsequence is explained in our next result.

For the rest of the section, we fix an invariant mean m on $l^\infty(G)$ and let M be the invariant mean on $l^\infty(G, \mathscr{L}(\mathfrak{H}))$ induced by m.

Proposition 6. *Let $h\colon G \to G$ be a homomorphism and let f be a function in $l^\infty(G, \mathscr{L}(\mathfrak{H}))$. Let $(k_i)_{i=1}^I$ be a finite K-valued sequence and let $(k_{i_j})_{j=1}^J$ be the selector subsequence associated with it. Then*

$$M^{\otimes I}_{(a_1,\dots,a_I)}\Big[f\Big(\sum_{i=1}^{I} k_i h(a_i)\Big)\Big] = M^{\otimes J}_{(a_1,\dots,a_J)}\Big[f\Big(\sum_{j=1}^{J} k_{i_j} h(a_j)\Big)\Big].$$

Proof. We proceed by induction on the length of $(k_i)_{i=1}^I$. If $(k_i)_{i=1}^I$ has only one element, then $(k_i)_{i=1}^I$ coincides with its associated selector subsequence and the assertion in this case holds vacuously for any function f in $l^\infty(G, \mathscr{L}(\mathfrak{H}))$. Suppose that the assertion holds for any K-valued sequence $(k_i)_{i=1}^I$ with a particular value of I and any function f in $l^\infty(G, \mathscr{L}(\mathfrak{H}))$. We shall show that it also holds for any K-valued sequence $(k_i)_{i=1}^{I+1}$ and any function f in $l^\infty(G, \mathscr{L}(\mathfrak{H}))$. Fix one such choice of $(k_i)_{i=1}^{I+1}$ and f arbitrarily. Let $(k_{i_j})_{j=1}^J$ be the selector subsequence associated with $(k_i)_{i=1}^I$. We consider two cases.

Assume first that $k_{I+1} = k_{i_l}$ for some $1 \leq l \leq J$. Then $(k_{i_j})_{j=1}^J$ is also the selector subsequence associated with $(k_i)_{i=1}^{I+1}$. It now suffices to show that

$$M^{\otimes (I+1)}_{(a_1,\dots,a_{I+1})}\Big[f\Big(\sum_{i=1}^{I+1} k_i h(a_i)\Big)\Big] = M^{\otimes I}_{(a_1,\dots,a_I)}\Big[f\Big(\sum_{i=1}^{I} k_i h(a_i)\Big)\Big]. \tag{3.1}$$

For any $a_1, \dots, a_{I+1} \in G$,

$$\begin{aligned} f\big(\sum_{i=1}^{I+1} k_i h(a_i)\big) \\ &= f\big(\sum_{i=1}^{k_{i_l}-1} k_i h(a_i) + k_{i_l} h(a_{i_l}) + \sum_{i=k_{i_l}+1}^{I} k_i h(a_i) + k_{I+1} h(a_{I+1})\big) \\ &= f\big(\sum_{i=1}^{k_{i_l}-1} k_i h(a_i) + k_{i_l} h(a_{i_l} + a_{I+1}) + \sum_{i=k_{i_l}+1}^{I} k_i h(a_i)\big), \end{aligned}$$

and so the function

$$(a_1, \dots, a_I) \mapsto f\big(\sum_{i=1}^{I+1} k_i h(a_i)\big)$$

is the translate of the function

$$(a_1, \dots, a_I) \mapsto f\big(\sum_{i=1}^{I} k_i h(a_i)\big)$$

by $(0, \dots, 0, a_{I+1}, 0, \dots, 0)$, with a_{I+1} precisely at the i_lth slot. Therefore, for each $a_{I+1} \in G$,

$$M^{\otimes I}_{(a_1,\dots,a_I)}\Big[f\big(\sum_{i=1}^{I+1} k_i h(a_i)\big)\Big] = M^{\otimes I}_{(a_1,\dots,a_I)}\Big[f\big(\sum_{i=1}^{I} k_i h(a_i)\big)\Big].$$

Applying M with respect to the variable a_{I+1} to both sides of the above equation and taking into account that the right-hand-side expression does not depend on a_{I+1}, we get

$$M_{a_{I+1}}\Big[M^{\otimes I}_{(a_1,\dots,a_I)}\Big[f\big(\sum_{i=1}^{I+1} k_i h(a_i)\big)\Big]\Big] = M^{\otimes I}_{(a_1,\dots,a_I)}\Big[f\big(\sum_{i=1}^{I} k_i h(a_i)\big)\Big].$$

Now equality (3.1) follows upon noting that, in view of Lemma 1,

$$M_{a_{I+1}}\Big[M^{\otimes I}_{(a_1,\dots,a_I)}\Big[f\big(\sum_{i=1}^{I+1} k_i h(a_i)\big)\Big]\Big] = M^{\otimes (I+1)}_{(a_1,\dots,a_{I+1})}\Big[f\big(\sum_{i=1}^{I+1} k_i h(a_i)\big)\Big].$$

Assume at present that k_{I+1} is different from any k_{i_l}, $1 \le l \le J$. Then the selector subsequence associated with $(k_i)_{i=1}^{I+1}$ has $J+1$ elements and is the extension of $(k_{i_j})_{j=1}^{J}$ with k_{I+1} taken as the last element. For each $a \in G$, let

$$g_a = T_{k_{I+1} h(a)} f.$$

Then, for any $a_1, \dots, a_{I+1} \in G$,

$$f\big(\sum_{i=1}^{I+1} k_i h(a_i)\big) = g_{a_{I+1}}\big(\sum_{i=1}^{I} k_i h(a_i)\big),$$

and so, for any $a_{I+1} \in G$,

$$M^{\otimes I}_{(a_1,\dots,a_I)}\Big[f\big(\sum_{i=1}^{I+1} k_i h(a_i)\big)\Big] = M^{\otimes I}_{(a_1,\dots,a_I)}\Big[g_{a_{I+1}}\big(\sum_{i=1}^{I} k_i h(a_i)\big)\Big].$$

By the inductive hypothesis applied to the function $g_{a_{I+1}}$,

$$M^{\otimes I}_{(a_1,\dots,a_I)}\Big[g_{a_{I+1}}\big(\sum_{i=1}^{I} k_i h(a_i)\big)\Big] = M^{\otimes J}_{(a_1,\dots,a_J)}\Big[g_{a_{I+1}}\big(\sum_{j=1}^{J} k_{i_j} h(a_j)\big)\Big].$$

Consequently,

$$M^{\otimes I}_{(a_1,\dots,a_I)}\Big[f\big(\sum_{i=1}^{I+1} k_i h(a_i)\big)\Big] = M^{\otimes J}_{(a_1,\dots,a_J)}\Big[g_{a_{I+1}}\big(\sum_{j=1}^{J} k_{i_j} h(a_j)\big)\Big].$$

Applying M with respect to the variable a_{I+1} to both sides of this equation, we obtain

$$M_{a_{I+1}}\Big[M^{\otimes I}_{(a_1,\dots,a_I)}\Big[f\big(\sum_{i=1}^{I+1} k_i h(a_i)\big)\Big]\Big] = \\ M_{a_{I+1}}\Big[M^{\otimes J}_{(a_1,\dots,a_J)}\Big[g_{a_{I+1}}\big(\sum_{j=1}^{J} k_{i_j} h(a_j)\big)\Big]\Big].$$

Upon replacing the dummy variable a_{I+1} by the variable a_{J+1} in the right-hand side expression, the last relation can be restated as

$$M_{a_{I+1}}\Big[M^{\otimes I}_{(a_1,\dots,a_I)}\Big[f\big(\sum_{i=1}^{I+1} k_i h(a_i)\big)\Big]\Big] = \\ M_{a_{J+1}}\Big[M^{\otimes J}_{(a_1,\dots,a_J)}\Big[g_{a_{J+1}}\big(\sum_{j=1}^{J} k_{i_j} h(a_j)\big)\Big]\Big].$$

Remembering that $k_{i_{J+1}} = k_{I+1}$ so that

$$g_{a_{J+1}}\big(\sum_{j=1}^{J} k_{i_j} h(a_j)\big) = f\big(\sum_{j=1}^{J+1} k_{i_j} h(a_j)\big),$$

we see that

$$M_{a_{J+1}}\Big[M^{\otimes J}_{(a_1,\dots,a_J)}\Big[g_{a_{J+1}}\big(\sum_{j=1}^{J} k_{i_j} h(a_j)\big)\Big]\Big] = \\ M_{a_{J+1}}\Big[M^{\otimes J}_{(a_1,\dots,a_J)}\Big[f\big(\sum_{j=1}^{J+1} k_{i_j} h(a_j)\big)\Big]\Big].$$

Thus

$$M_{a_{I+1}}\Big[M^{\otimes I}_{(a_1,\dots,a_I)}\Big[f\Big(\sum_{i=1}^{I+1} k_i h(a_i)\Big)\Big]\Big] = M_{a_{J+1}}\Big[M^{\otimes J}_{(a_1,\dots,a_J)}\Big[f\Big(\sum_{j=1}^{J+1} k_{i_j} h(a_j)\Big)\Big]\Big],$$

which, in view of Lemma 1, is equivalent to

$$M^{\otimes I+1}_{(a_1,\dots,a_{I+1})}\Big[f\Big(\sum_{i=1}^{I+1} k_i h(a_i)\Big)\Big] = M^{\otimes J+1}_{(a_1,\dots,a_{J+1})}\Big[f\Big(\sum_{j=1}^{J+1} k_{i_j} h(a_j)\Big)\Big],$$

as was to be shown. □

3.3. Operators with non-negative spectrum

Let $\boldsymbol{A}$ be a complex unital Banach algebra. For an element a of $\boldsymbol{A}$, denote by $\sigma_{\boldsymbol{A}}(a)$ the spectrum of a relative to $\boldsymbol{A}$. An element a of $\boldsymbol{A}$ is said to have *non-negative spectrum* relative to $\boldsymbol{A}$ if $\sigma_{\boldsymbol{A}}(a) \subset [0,\infty)$. If $\boldsymbol{A}$ is commutative, then an equivalent condition for $a \in \boldsymbol{A}$ to have non-negative spectrum relative to $\boldsymbol{A}$ is that $\phi(a) \geq 0$ for each $\phi \in \Delta(\boldsymbol{A})$. This follows from the fact that $\sigma_{\boldsymbol{A}}(a) = \{\phi(a) \mid \phi \in \Delta(\boldsymbol{A})\}$ (see e.g. [2, Chapter 1, §16, Proposition 9]).

Let $u\colon G \to \mathscr{L}(\mathfrak{H})$ be a uniformly bounded K-spherical function and let $h\colon G \to G$ be a homomorphism. Denote by $\mathscr{P}_0(K)$ the set of all non-empty subsets of K. For a set A in $\mathscr{P}_0(K)$, let S_A be the set of all bijective sequences with values in A; in other words, the members of S_A is are bijections of the form $s_A\colon \{1,\dots,|A|\} \to A$. For each $A \in \mathscr{P}_0(K)$ and each $s_A \in S_A$, let

$$P_{s_A}(u,h) = M^{\otimes |A|}_{(a_1,\dots,a_{|A|})}\Big[u\Big(\sum_{i=1}^{|A|} s_A(i) h(a_i)\Big)\Big].$$

Where there is no ambiguity, as it will be the case in the rest of this section, we abbreviate $P_{s_A}(u,h)$ to P_{s_A}. In view of Proposition 2 and Lemma 2(iii), the family

$$\{P_{s_A} \mid A \in \mathscr{P}_0(K),\ s_A \in S_A\}$$

is commutative. The following result will be instrumental in the proof of Theorem 1.

Proposition 7. *Let $\boldsymbol{A}$ be a unital commutative Banach subalgebra of $\mathscr{L}(\mathfrak{H})$ containing $\{P_{s_A} \mid A \in \mathscr{P}_0(K),\ s_A \in S_A\}$. Then, for each $A \in \mathscr{P}_0(K)$ and each $s_A \in S_A$, P_{s_A} has non-negative spectrum relative to $\boldsymbol{A}$.*

Proof. Let $A, B \in \mathscr{P}_0(K)$, and let $s_A \in S_A$ and $s_B \in S_B$. Then, by Lemma 2(i),

$$\begin{aligned} &P_{s_A} P_{s_B} \\ &= |K|^{-1} \sum_{k \in K} M^{\otimes(|A|+|B|)}_{(a_1,\dots,a_{|A|},b_1,\dots,b_{|B|})} \Big[u\Big(\sum_{i=1}^{|A|} s_A(i) h(a_i) + k\big(\sum_{j=1}^{|B|} s_B(j) h(b_j) \big) \Big) \Big] \\ &= |K|^{-1} \sum_{k \in K} M^{\otimes(|A|+|B|)}_{(a_1,\dots,a_{|A|},b_1,\dots,b_{|B|})} \Big[u\big(\sum_{i=1}^{|A|} s_A(i) h(a_i) + \sum_{j=1}^{|B|} \mathring{s}_{kB}(j) h(b_j) \big) \Big], \end{aligned} \tag{3.2}$$

where, for each $k \in K$, $\mathring{s}_{kB} = \mathring{s}_{kB}(k, s_B)$ denotes the sequence in S_{kB} given by $\mathring{s}_{kB}(i) = k s_B(i)$ for each $i = 1, \dots, |B|$; note that B and kB have equal cardinality. Let $s_A \oplus \mathring{s}_{kB}$ be the concatenation of the sequences s_A and $\mathring{s}_{kB}$ defined by

$$(s_A \oplus \mathring{s}_{kB})(i) = \begin{cases} s_A(i) & \text{if } 1 \le i \le |A|, \\ \mathring{s}_{kB}(i - |A|) & \text{if } |A| < i \le |A| + |B|. \end{cases}$$

The range of $s_A \oplus \mathring{s}_{kB}$ is precisely $A \cup kB$. It is obvious that, for each $k \in K$,

$$\begin{aligned} &M^{\otimes(|A|+|B|)}_{(a_1,\dots,a_{|A|},b_1,\dots,b_{|B|})} \Big[u\big(\sum_{i=1}^{|A|} s_A(i) h(a_i) + \sum_{j=1}^{|B|} \mathring{s}_{kB}(j) h(b_j) \big) \Big] \\ &\qquad = M^{\otimes(|A|+|B|)}_{(a_1,\dots,a_{|A|+|B|})} \Big[u\big(\sum_{i=1}^{|A|+|B|} (s_A \oplus \mathring{s}_{kB})(i) h(a_i) \big) \Big]. \end{aligned} \tag{3.3}$$

A moment's reflection reveals that the selector subsequence associated with $s_A \oplus \mathring{s}_{kB}$ is the sequence $\overset{+}{s}_{A \cup kB} = \overset{+}{s}_{A \cup kB}(s_A, \mathring{s}_{kB})$ in $S_{A \cup kB}$ given by

$$\overset{+}{s}_{A \cup kB}(l) = \begin{cases} s_A(l) & \text{if } 1 \le l \le |A|, \\ \mathring{s}_{kB}(i_l) & \text{if } |A| < l \le |A \cup kB|, \end{cases} \tag{3.4}$$

where i_l is the smallest j for which $\mathring{s}_{kB}(j)$ is in $kB \setminus (A \cup \{\mathring{s}_{kB}(i_{|A|+1}), \dots, \mathring{s}_{kB}(i_{l-1})\})$. In view of Proposition 6,

$$\begin{aligned} &M^{\otimes(|A|+|B|)}_{(a_1,\dots,a_{|A|+|B|})} \Big[u\big(\sum_{i=1}^{|A|+|B|} (s_A \oplus \mathring{s}_{kB})(i) h(a_i) \big) \Big] \\ &\qquad = M^{\otimes(|A \cup kB|)}_{(a_1,\dots,a_{|A \cup kB|})} \Big[u\big(\sum_{i=1}^{|A \cup kB|} \overset{+}{s}_{A \cup kB}(i) h(a_i) \big) \Big]. \end{aligned}$$

It now follows from (3.2) and (3.3) that

$$P_{s_A} P_{s_B} = |K|^{-1} \sum_{k \in K} P_{\overset{+}{s}_{A \cup kB}}. \tag{3.5}$$

This identity will play a crucial role in the rest of the proof.

To show that for each $A \in \mathscr{P}_0(K)$ and each $s_A \in S_A$, P_{s_A} has non-negative spectrum relative to $\boldsymbol{A}$, we proceed by induction on the cardinality of A in descending order. Let s_K be any sequence in S_K. For each $k \in K$, we have $K = kK$ and further $\overset{+}{s}_{K \cup kK} = s_K$, as is immediately clear from (3.4). Thus, in view of (3.5), $P_{s_K}^2 = P_{s_K}$. Now, if $\phi \in \Delta(\boldsymbol{A})$, then $(\phi(P_{s_K}))^2 = \phi(P_{s_K}^2) = \phi(P_{s_K})$, and so $\phi(P_{s_K})$ is equal either to 0 or 1. This shows that P_{s_K} has non-negative spectrum relative to $\boldsymbol{A}$.

Suppose now that the assertion is true for each $A \in \mathscr{P}_0(K)$ with $|A| \geq c$, where $2 \leq c \leq |K|$, and each $s_A \in S_A$. We shall show that the assertion is also true for any $A \in \mathscr{P}_0(K)$ with $|A| = c - 1$ and any $s_A \in S_A$. Fix $A \in \mathscr{P}_0(K)$ with $|A| = c - 1$ and $s_A \in S_A$ arbitrarily. Let $\phi \in \Delta(\boldsymbol{A})$. The rest of the argument will consist of the verification that $\phi(P_{s_A}) \geq 0$.

We first demonstrate that $\phi(P_{s_A} P_{s_B}) \geq 0$ for each $B \in \mathscr{P}_0(K)$ with $|B| \geq c$ and each $s_B \in S_B$. Given any such B and s_B, note that, for each $k \in K$, we have $|A \cup kB| \geq c$, as $|kB| = |B|$. By the inductive hypothesis, for each $k \in K$, $P_{\overset{+}{s}_{A \cup kB}}$ has non-negative spectrum and in particular $\phi(P_{\overset{+}{s}_{A \cup kB}}) \geq 0$. Consequently, in view of (3.5),

$$\phi(P_{s_A} P_{s_B}) = |K|^{-1} \sum_{k \in K} \phi(P_{\overset{+}{s}_{A \cup kB}}) \geq 0,$$

as desired.

We now consider two cases. Suppose first that there exists $B \in \mathscr{P}_0(K)$ with $|B| \geq c$ and $P_{s_B} \in S_B$ such that $\phi(P_{s_B}) \neq 0$. By the inductive hypothesis, we then necessarily have $\phi(P_{s_B}) > 0$. By the previous paragraph, $\phi(P_{s_A} P_{s_B}) \geq 0$. But $\phi(P_{s_A})\phi(P_{s_B}) = \phi(P_{s_A} P_{s_B})$, and so $\phi(P_{s_A}) \geq 0$.

Suppose now that $\phi(P_{s_B}) = 0$ whenever $B \in \mathscr{P}_0(K)$ satisfies $|B| \geq c$ and $s_B \in S_B$. Let K_A be the set of those $k \in K$ for which $A \cup kA = A$. Incidentally, this set is identical with the stabiliser of A, that is, the set of those $k \in K$ for which $kA = A$; indeed, $A \cup kA = A$ means that $kA \subset A$, but as kA and A are finite and of equal cardinality, the containment $kA \subset A$ holds precisely when the equality $kA = A$ holds. The set K_A is non-empty as it contains e, and in fact it forms a subgroup of K. If $k \in K_A$, then, as is immediately seen from (3.4), we have $\overset{+}{s}_{A \cup kA} = s_A$ and in particular $\phi(P_{\overset{+}{s}_{A \cup kA}}) = \phi(P_{s_A})$. If $k \in K \setminus K_A$, then A is properly contained in $A \cup kA$, implying that $|A \cup kA| \geq c$, and further, by the current assumption, that $\phi(P_{\overset{+}{s}_{A \cup kA}}) = 0$. Thus, on account of (3.5),

$$\phi(P_{s_A})^2 = \phi(P_{s_A}^2) = |K|^{-1} \sum_{k \in K} \phi(P_{\overset{+}{s}_{A \cup kA}}) = |K|^{-1} |K_A| \phi(P_{s_A}),$$

whence $\phi(P_A)$ is equal either to 0 or $|K|^{-1}|K_A|$. This proves the inductive step and the proposition. $\square$

4. Proof of Theorem 1

In this section we give a proof of the main result of the paper.

Proof of Theorem 1. Let m be an invariant mean on $l^\infty(G)$. Let $\langle\!\langle \cdot,\cdot \rangle\!\rangle$ be the semi-inner product in $\mathfrak{H}$ defined by

$$\langle\!\langle x, y \rangle\!\rangle = m_a(\langle u(a)x, u(a)y \rangle) \quad (x, y \in \mathfrak{H}),$$

and let $|||\cdot|||$ be the semi-norm induced by this semi-inner product; that is,

$$|||x||| = (m_a(\|u(a)x\|^2))^{1/2} \quad (x \in \mathfrak{H}).$$

For any $a \in G$ and any $x, y \in \mathfrak{H}$, we have

$$\langle\!\langle u(a)x, y \rangle\!\rangle = \langle\!\langle x, u(-a)y \rangle\!\rangle. \tag{4.1}$$

Indeed,

$$\begin{aligned}
\langle\!\langle u(a)x, y \rangle\!\rangle &= m_b(\langle u(b)u(a)x, u(b)y \rangle) \\
&= \frac{1}{|K|} \sum_{k \in K} m_b(\langle u(b+ka)x, u(b)y \rangle) \\
&= \frac{1}{|K|} \sum_{k \in K} m_b(\langle u(b)x, u(b-ka)y \rangle) \\
&= m_b(\langle u(b)x, u(b)u(-a)y \rangle) \\
&= \langle\!\langle x, u(-a)y \rangle\!\rangle,
\end{aligned}$$

where the third equality follows from the invariance property of m and the fact that, for each $k \in K$, $b \mapsto \langle u(b)x, u(b-ka)y \rangle$ is the translate of $b \mapsto \langle u(b+ka)x, u(b)y \rangle$ by $-ka$.

The mapping $(x, y) \mapsto \langle\!\langle x, y \rangle\!\rangle$ is a positive semi-definite bounded sesquilinear form on $\mathfrak{H} \times \mathfrak{H}$. By the Riesz representation theorem, there exists a unique positive operator R in $\mathscr{L}(\mathfrak{H})$ such that

$$\langle\!\langle x, y \rangle\!\rangle = \langle Rx, y \rangle \quad (x, y \in \mathfrak{H}).$$

Let S be the unique positive square root of R. Then

$$\langle\!\langle x, y \rangle\!\rangle = \langle Sx, Sy \rangle \quad (x, y \in \mathfrak{H}), \tag{4.2}$$

or equivalently

$$|||x||| = \|Sx\| \quad (x \in \mathfrak{H}). \tag{4.3}$$

It is clear that $|||x||| \le \|u\|_\infty \|x\|$ for each $x \in \mathfrak{H}$, so

$$\|S\| \le \|u\|_\infty. \tag{4.4}$$

Let M be the invariant mean on $l^\infty(G, \mathscr{L}(\mathfrak{H}))$ induced by m (and constructed with the aid of $\langle \cdot,\cdot \rangle$). For each $k \in K$, define P_k in $\mathscr{L}(\mathfrak{H})$ by

$$P_k = M_a(u(a - ka)).$$

Note that $P_e = \mathrm{id}_{\mathfrak{H}}$. For each $k \in K$, let $h_k \colon G \to G$ be the homomorphism given by

$$h_k(a) = a - ka \quad (a \in G)$$

and, consistently with the notation of Subsection 3.3, let $s_{\{k\}}$ denote the single-element sequence containing k as its only element. Using the terminology from Subsection 3.3, we have

$$P_k = P_{s_{\{k\}}}(u, h_k) \quad (k \in K).$$

In view of Proposition 2 and Lemma 2(iii), the family

$$\{P_{s_A}(u, h_k) \mid k \in K,\ A \in \mathscr{P}_0(K),\ s_A \in S_A\}$$

is commutative. Let $\boldsymbol{A}$ be the unital commutative Banach subalgebra of $\mathscr{L}(\mathfrak{H})$ generated by this family. By Proposition 7, each $P_{s_A}(u, h_k)$ and in particular each P_k has non-negative spectrum relative to $\boldsymbol{A}$. Let $T \in \mathscr{L}(\mathfrak{H})$ be defined by

$$T = M_a(u(a)u(-a)).$$

Since, for each $a \in G$,

$$u(a)u(-a) = \frac{1}{|K|}\Bigg(\mathrm{id}_{\mathfrak{H}} + \sum_{k \in K \setminus \{e\}} u(a - ka)\Bigg),$$

it follows that

$$T = \frac{1}{|K|}\Bigg(\mathrm{id}_{\mathfrak{H}} + \sum_{k \in K \setminus \{e\}} P_k\Bigg).$$

For each $\phi \in \Delta(\boldsymbol{A})$, we have $\phi(P_k) \geq 0$ for each $k \in K \setminus \{e\}$ by Proposition 7, and also $\phi(\mathrm{id}_{\mathfrak{H}}) = 1$, which implies that $\phi(T) \geq |K|^{-1}$. In particular, $\phi(T)$ is non-zero for each $\phi \in \Delta(\boldsymbol{A})$, and so T is invertible in $\boldsymbol{A}$. Now, using the fact that $u(a)$ and $u(-a)$ commute for each $a \in G$ and the elementary fact that every sesquilinear positive semi-definite form satisfies the Cauchy–Schwartz inequality, we deduce that

$$\begin{aligned}
\|x\|^2 &= \langle Tx, (T^{-1})^* x\rangle = m_a(\langle u(-a)u(a)x, (T^{-1})^* x\rangle) \\
&\leq m_a(\|u(-a)u(a)x\|)\, \|(T^{-1})^* x\| \\
&\leq \|u\|_\infty\, (m_a(\|u(a)x\|^2)^{1/2}\, \|(T^{-1})^*\| \|x\| \\
&= \|u\|_\infty \|\|x\|\| \|T^{-1}\| \|x\|,
\end{aligned}$$

whence, on account of (4.3),

$$\|x\| \leq \|u\|_\infty \|T^{-1}\| \|\|x\|\| = \|u\|_\infty \|T^{-1}\| \|Sx\|.$$

Given that S is positive and as such hermitian, this relation entails that S has an inverse in $\mathscr{L}(\mathfrak{H})$ (cf. [13, Theorem 3.2.6]).

Consider now the K-spherical function $\tilde{u}\colon G \to \mathscr{L}(\mathfrak{H})$ defined by

$$\tilde{u}(a) = Su(a)S^{-1} \quad (a \in G).$$

It follows from (4.1) and (4.2) that $\tilde{u}$ is a $*$-mapping. In view of (4.4), to complete the proof it suffices to show that

$$\|S^{-1}\| \leq |K| \|u\|_\infty. \tag{4.5}$$

Note that, by Proposition 3, $\tilde{u}$ is normal. Let $\boldsymbol{C}$ be the C^*-algebra subalgebra of $\mathscr{L}(\mathfrak{H})$ generated by $\{\tilde{u}(a) \mid a \in G\}$. In view of the Fuglede–Putnam–Rosenblum theorem, $\boldsymbol{C}$ is commutative. Let $\boldsymbol{W}$ be the von Neumann algebra generated by $\boldsymbol{C}$. Since taking the weak-operator closure preserves commutativity, $\boldsymbol{W}$ is commutative too. For each $k \in K$, define $\tilde{P}_k$ in $\mathscr{L}(\mathfrak{H})$ by

$$\tilde{P}_k = M_a(\tilde{u}(a - ka)).$$

In the terminology of Subsection 3.3, we have

$$\tilde{P}_k = P_{s_{\{k\}}}(\tilde{u}, h_k) \quad (k \in K).$$

It is readily seen that the family

$$\{P_{s_A}(\tilde{u}, h_k) \mid k \in K,\ A \in \mathscr{P}_0(K),\ s_A \in S_A\}$$

is contained in $\boldsymbol{W}$. By Proposition 7, each $P_{s_A}(\tilde{u}, h_k)$ and in particular each $\tilde{P}_k$ has non-negative spectrum relative to $\boldsymbol{W}$. Since any operator P in $\mathscr{L}(\mathfrak{H})$ has the same spectrum relative to all sub-C^*-algebras of $\mathscr{L}(\mathfrak{H})$ that contain P (cf. [17, Theorem 11.29]), it follows that $\tilde{P}_k$ is a positive operator in $\mathscr{L}(\mathfrak{H})$.

Let $\tilde{T} \in \mathscr{L}(\mathfrak{H})$ be defined by

$$\tilde{T} = M_a(\tilde{u}(a)\tilde{u}(-a)).$$

Then

$$\tilde{T} = \frac{1}{|K|}\left(\mathrm{id}_{\mathfrak{H}} + \sum_{k \in K \setminus \{e\}} \tilde{P}_k\right)$$

and consequently

$$|K|^{-1}\|x\|^2 \le \langle \tilde{T}x, x\rangle \tag{4.6}$$

for each $x \in \mathfrak{H}$. Taking into account that $\tilde{u}(a)\tilde{u}(-a) = Su(a)u(-a)S^{-1} = Su(-a)u(a)S^{-1}$ for each $a \in G$, we see that

$$\langle \tilde{T}x, x\rangle = m_a(\langle Su(-a)u(a)S^{-1}x, x\rangle).$$

Bearing in mind that S is hermitian, we have

$$\begin{aligned}\langle Su(-a)u(a)S^{-1}x, x\rangle &= \langle u(-a)u(a)S^{-1}x, Sx\rangle \\ &\le \|u(-a)u(a)S^{-1}x\|\|Sx\| \\ &\le \|u\|_\infty \|u(a)S^{-1}x\|\|Sx\|.\end{aligned}$$

Now, in view of (4.3) and the Cauchy–Schwartz inequality,

$$\begin{aligned}\langle \tilde{T}x, x\rangle &\le \|u\|_\infty m_a(\|u(a)S^{-1}x\|)\|Sx\| \\ &\le \|u\|_\infty (m_a(\|u(a)S^{-1}x\|^2)^{1/2}\|Sx\| \\ &= \|u\|_\infty \|\|S^{-1}x\|\|\|Sx\| \\ &= \|u\|_\infty \|x\|\|Sx\|.\end{aligned}$$

Combining this with (4.6), we find that

$$\|x\|^2 \le |K|\|u\|_\infty \|x\|\|Sx\|$$

and further

$$\|x\| \le |K| \|u\|_\infty \|Sx\|.$$

Substituting $S^{-1}x$ for x finally yields

$$\|S^{-1}x\| \le |K| \|u\|_\infty \|x\|,$$

and this establishes (4.5). □

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